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Relation between Charge and Energy Conservation in a Nonlinear Electrodynamics

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A new mathematical formulation of electrodynamics is presented in which the field equations and the conservation law for the energy-momentum tensor appear as the components of a single geometric object. The construction is based upon a geometric structure on the 2-forms over an even-dimensional vector space that parallels a geometric structure on 1-forms over $\mathbb{R}^4$ determined by special relativity. In this construction charge appears as the analog of mass. In special relativity the conservation of mass implies the relation $(d/dt)e = \langle f, v\rangle$; here the conservation of charge implies the relation $\operatorname{div} E = i(J)F$, when the energy-momentum tensor E and field strength F are given a "relativistic" interpretation.

1. INTRODUCTION

I present a model of electrodynamics that leads to a unification of the equations of motion and the conservation law that is both formally and mathematically similar to the unification of power and force obtained in relativistic mechanics. One advantage of relativistic mechanics is that it allows a deeper understanding of the relation between the conservation of energy and the conservation of mass. It does so by modifying the conservation law so that the classical relation is obtained in the limit of small velocities. The mathematical technique that underlies the construction of relativistic mechanics is actually a special and easy example of an extended dynamical formalism. By applying this formalism to electromagnetism, I develop a nonlinear electrodynamics that is formally analogous to relativistic mechanics. A complete description of the general construction will appear elsewhere.

Classical mechanics fails to provide a satisfactory representation of the relation between energy and mass until one gives up the principle of universal time; that is, the parametrizations of solutions to a given dynamical

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problem cannot be fixed *a priori*, but depend upon the nature of the problem. The premise of this article is that a similar prescription applies to electrodynamics. What is given up is the representation of the vector potential by a 1-form. What is gained is a nonlinear electrodynamics that relates charge conservation to the conservation of field energy-momentum. However, as in special relativity, the classical laws are obtained only as approximations to the complete "relativistic" expression in the limit of small field strengths.

The remainder of this article is divided into three sections. Section 2 develops a linear algebraic structure on the space of 2-forms over an even-dimensional vector space that has properties similar to the conformal Lorentz group. In Section 3 the results of Section 2 are used to derive dynamical equations for electromagnetism that are similar to those obtained by Born and Infeld (1934). Concluding remarks are given in Section 4.

2. RELATIVISTIC STRUCTURES FOR 2-FORMS

This section develops a linear algebraic structure on the nondegenerate 2-forms over an even-dimensional vector space that parallels the relativistic structure on the 1-form over Minkowski space. Let V be a real vector space of dimension $2n$. If X is a subspace of V, denote the subspace of the dual space that annihilates X by $\mathrm{ann}(X)$. Recall that if $A \in \mathrm{End}(V)$ satisfies $\ker(A) = \ker(A^2)$, then A possesses a commutative generalized inverse, $A^{\#} \in \mathrm{End}(V)$. The $A^{\#}$ is uniquely determined by the relations (1) $A^{\#}AA^{\#} = A^{\#}$, (2) $AA^{\#}A = A$, (3) $A^{\#}A = AA^{\#}$. Also, if $A \in \mathrm{End}(V)$, denote the spectrum of A by $\mathrm{spec}(A)$, which is the set of eigenvalues of A, each occurring as many times as their multiplicity.

The primary geometric objects of this construction are ordered pairs of n-dimensional subspaces of V. A pair (X, Y) is a *splitting* of V if $X \cap Y = 0$. A splitting (X, Y) determines a projection $P \in \mathrm{End}(V)$ defined by setting $\mathrm{ran}(P) = Y$ and $\ker(P) = X$. The complementary projection $1 - P$ shall be denoted by $P^{\perp}$. Given two pairs of subspaces (X, Y) and (X', Y'), the ordered pair $((X, Y), (X', Y'))$ is called a *transverse pair* if $X \cap Y' = 0$ and $X' \cap Y = 0$. If (X, Y) is a splitting, a transverse pair $((X, Y), (X', Y'))$ determines a pair of linear maps (A^*, A); (the star has no algebraic significance). We define $A^* \in \mathrm{End}(V)$ by (1) $\mathrm{ran}(A^*) \subset Y \subset \ker(A^*)$ and (2) for any $u \in X$, $u + A^*u \in X'$. Similarly, $A \in \mathrm{End}(V)$ is defined by the conditions (1) $\mathrm{ran}(A) \subset X \subset \ker(A)$ and (2) for any $u \in Y$, $u + Au \in Y'$. The pair (A^*, A) is called the *graph coordinate* of $((X, Y), (X', Y'))$.

Proposition 2.1. Let (X, Y) be a splitting, and let $((X, Y), (X', Y'))$ be a transverse pair with graph coordinate (A^*, A). Now, (X', Y') is a splitting if $1 \notin \mathrm{spec}(A^*A)$, or equivalently, if $\mathrm{ran}(P - A^*A) = Y$, where P is the projection defined by (X, Y).

In the following it shall be assumed that if (A^*, A) is the graph coordinate of $((X, Y), (X', Y'))$, then $\mathrm{spec}(A^*A) \subset \mathbb{C} - [1, \infty)$ and $\mathrm{spec}(AA^*) \subset \mathbb{C} - [1, \infty)$. This assumption and Proposition 2.1 show that $(P - A^*A)^\#$ is well-defined. Let $g(z)$ be the interpolating polynomial on the nonzero spectrum of $(P - A^*A)^\#$ for the standard branch of $f(z) = z^{1/2}$. Since $\mathrm{spec}(P - A^*A) = \mathrm{spec}(P^\perp - AA^*)$, $g(z)$ can be used to define square roots for $(P - A^*A)^\#$ and $(P^\perp - AA^*)^\#$ given by $\gamma = g((P - A^*A)^\#)$ and $\gamma^* = g((P^\perp - AA^*)^\#)$. The following fact is useful in computations.

Proposition 2.2. $A\gamma = \gamma^* A$ and $\gamma A^* = A^* \gamma^*$.

Proof. Note that

$$\begin{aligned} A(P - A^*A)^\# &= (P^\perp - AA^*)^\#(P^\perp - AA^*)A(P - A^*A)^\# \\ &= (P^\perp - AA^*)^\# A(P - A^*A)(P - A^*A)^\# \\ &= (P^\perp - AA^*)^\# A \quad \blacksquare \end{aligned}$$

Now I introduce some terminology from symplectic geometry. An n-dimensional subspace X of V is a *Lagrangian subspace* relative to a nondegerate 2-form ω if for any $u, v \in X$, then $\omega(u, v) = 0$. A pair of subspaces (X, Y) is a *Lagrangian pair* relative to ω if X and Y are Lagrangian subspaces relative to ω. Background information on symplectic geometry can be found in Sternberg and Guillemin (1984) and Weinstein (1977). I also introduce the following notation. Let (X, Y) be a splitting of V. Let $D_{(X,Y)}$ be the set of nondegenerate 2-forms on V having (X, Y) as a Lagrangian pair. Similarly, viewing 2-vectors as forms on the dual of V, denote by $D^*_{(X,Y)}$ the set of nondegenerate 2-vectors on V having $(\mathrm{ann}(X), \mathrm{ann}(Y))$ as a Lagrangian pair. Let $M_{(X,Y)} \subset \mathrm{End}(V)$ be defined by $M_{(X,Y)} = \{E \mid \mathrm{ran}(E) = Y \text{ and } \ker(E) = X\}$, and let $V_{(X,Y)} \subset \mathrm{End}(V) \times \mathrm{End}(V)$ be the set of graph coordinates that satisfy the above restrictions. That is,

$$\begin{aligned} V_{(X,Y)} = \{(A^*, A) \mid\ & \mathrm{ran}(A^*) \subset Y \subset \ker(A^*), \mathrm{ran}(A) \subset X \subset \ker(A), \\ & \text{and } \mathrm{spec}(A^*A) \subset \mathbb{C} - [1, \infty)\} \end{aligned}$$

Definition 2.1. Let $((X, Y), (X', Y'))$ be a transverse pair of splittings with graph coordinate $(A^*, A) \in V_{(X,Y)}$. For $\omega \in D_{(X,Y)}$ and $\Lambda \in D^*_{(X,Y)}$, define $\omega_{(A^*,A)} \in D_{(X',Y')}$ and $\Lambda_{(A^*,A)} \in D^*_{(X',Y')}$ as follows. If $u, v \in V$, let

$$\begin{aligned} \omega_{(A^*,A)}(u, v) = \omega(\gamma(P - A^*)u, \gamma^*(P^\perp - A)v) \\ + \omega(\gamma^*(P^\perp - A)u, \gamma(P - A^*)v) \end{aligned} \tag{1}$$

and if $\lambda, \mu \in V^*$, let

$$\begin{aligned} \Lambda_{(A^*,A)}(\lambda, \mu) = \Lambda(((P + A)\gamma)^T\lambda, ((P^\perp + A^*)\gamma^*)^T\mu) \\ + \Lambda(((P^\perp + A^*)\gamma^*)^T\lambda, ((P + A)\gamma)^T\mu) \end{aligned} \tag{2}$$

Also, if $E \in M_{(X,Y)}$, define $\omega_E \in D_{(X,Y)}$ and $\Lambda_E \in D^*_{(X,Y)}$ as follows. If $u, v \in V$, let

$$\omega_E(u, v) = \omega(Eu, v) + \omega(u, Ev) \tag{3}$$

and if $\lambda, \mu \in V^*$, let

$$\Lambda_E(\lambda, \mu) = \Lambda(E^T\lambda, \mu) + \Lambda(\lambda, E^T\mu) \tag{4}$$

Definition 2.2. For any 2-form ω and 2-vector Λ on V the *charge* of the pair (Λ, ω) is the endomorphism $\mathscr{C}(\Lambda, \omega) \in \mathrm{End}(V)$ given by $\mathscr{C}(\Lambda, \omega) = C(\Lambda \otimes \omega)$, where C denotes the contraction on the second and third entries.

The reasons behind this choice of terminology will become apparent in the next section. Definition 2.1 and 2.2 have the following consequences. First note that a nondegenerate 2-form ω on V determines a transpose on $\mathrm{End}(V)$. For $A \in \mathrm{End}(V)$, define $A^t \in \mathrm{End}(V)$ for $u, v \in V$ by $\omega(u, Av) = \omega(A^t u, v)$.

Proposition 2.3. If $((X, Y), (X', Y'))$ is a transverse pair of splittings with graph coordinate $(A^*, A) \in V_{(X,Y)}$ and if (A'^*, A') are the graph coordinates of the pair $((X', Y'), (X, Y))$, then, for $\omega \in D_{(X,Y)}$ and $\Lambda \in D^*_{(X,Y)}$,

$$(\omega_{(A^*,A)})_{(A'^*,A')} = \omega, \qquad (\Lambda_{(A^*,A)})_{(A'^*,A')} = \Lambda.$$

Proposition 2.4. If $((X, Y), (X', Y'))$ is a transverse pair of splittings with graph coordinate (A^*, A) and $\omega \in D_{(X,Y)}$ and $\Lambda \in D^*_{(X,Y)}$, then (1)

$$\begin{aligned}\mathscr{C}(\Lambda_{(A^*,A)}, \omega_{(A^*,A)}) = {} & (P^\perp + A^*)\gamma^*\mathscr{C}(\Lambda, \omega)\gamma^*(P^\perp - A) \\ & + (P + A)\gamma\mathscr{C}(\Lambda, \omega)\gamma(P - A^*)\end{aligned}$$

and (2) if $E \in M_{(X,Y)}$ and if the transpose is determined by ω, then

$$\mathscr{C}(\Lambda_E, \omega_E) = E\mathscr{C}(\Lambda, \omega)E + E\mathscr{C}(\Lambda, \omega)E^t + \mathscr{C}(\Lambda, \omega)E^t E^t$$

The proofs of Proposition 2.3 and Proposition 2.4 are rather long computations that follow from Definition 2.1. They can also be formulated in group-theoretic language. More on this aspect of this construction shall appear later.

Propositions 2.3 and 2.4 indicate that the identities (1)-(4) possess properties similar to the correspondence between velocity and mass, and momentum in special relativity. In fact, if $\mathscr{C}(\Lambda, \omega) = cI$, then $\mathscr{C}(\Lambda_{(A^*,A)}, \omega_{(A^*,A)}) = cI$ and $\mathscr{C}(\Lambda_E, \omega_E) = cE^2 + cE^{t2}$. Thus, under transformations (1) and (2) a scalar charge is invariant, while transformations (3) and (4) scale the charge by $E^2 + E^{t2}$. These results are extensions of the familiar facts from special relativity that mass is an invariant of the Lorentz group but is variant under scale transformations.

The following propositions show that the structure of the transformations (1)–(4) is somewhat more complicated than the structure of the conformal Lorentz group. Since (2) and (4) are dual to (1) and (3), attention shall be restricted to 2-forms. First I introduce some terminology. Given a nondegenerate 2-form ω, $A \in \mathrm{End}(V)$ is symmetric relative to ω if $A^t = -A$, and A is skew symmetric if $A^t = A$. The switch in sign from the usual usage of these terms is due to the fact that the transpose is defined relative to a skew-symmetric form.

Proposition 2.5. Let $((X, Y), (X', Y'))$ be a transverse pair of splittings with graph coordinate $(A^*, A) \in V_{(X,Y)}$ and let $\omega \in D_{(X,Y)}$. Then $\omega_{(A^*,A)} = \omega$ if and only if A^* and A are symmetric.

Proof. If $\omega_{(A^*,A)} = \omega$, then for $u, v \in X$, $\omega(\gamma A^* u, \gamma^* v) + \omega(\gamma^* u, \gamma A^* v) = 0$ and for $u, v \in Y$, $\omega(\gamma u, \gamma^* A v) + \omega(\gamma^* A u, \gamma v) = 0$. These relations and Proposition 2.2 imply that A^* and A are symmetric. If A^* and A are symmetric, then $(P - A^*A)^t = P^\perp - AA^*$ and so $\gamma^t = \gamma^*$. Therefore, if $B = P^\perp - A$ and $C = P^\perp + A^*$, then for $u, v \in V$,

$$\omega_{(A^*,A)}(u, v) = \omega(u, C(BC)^\# Bv) + \omega(C(BC)^\# Bu, v)$$

But

$$C(BC)^\# B = CB(CB)^{\#2} CB = CB(CB)^\# = P'$$

where P' is the projection determined by the pair (Y', X'). Since X' and Y' are Lagrangian subspaces for ω, it follows that $\omega_{(A^*,A)} = \omega$. ∎

Corollary 2.1. Let $\omega \in D_{(X,Y)}$ and let $(A^*, A) \in V_{(X,Y)}$. If X and Y are Lagrangian subspaces for $\omega_{(A^*,A)}$, then $\omega_{(A^*,A)} = \omega$.

Proof. If X and Y are Lagrangian subspaces for $\omega_{(A^*,A)}$ then A^* and A are symmetric. ∎

The next proposition gives the results of some simple computations involving Definition 2.1 that are used in the following arguments.

Proposition 2.6. Let $((X, Y), (X', Y'))$ be a transverse pair with graph coordinate $(A^*, A) \in V_{(X,Y)}$ and let $\omega \in D_{(X,Y)}$. (1) If $E \in M_{(X,Y)}$ and if $E' = (P + A)\gamma E \gamma (P - A^*)$, then $E' \in M_{(X',Y')}$ and $(\omega_E)_{(A^*,A)} = (\omega_{(A^*,A)})_{E'}$. (2) Let $(C^*, C) \in V_{(X,Y)}$. If $C'^* = (P + A)\gamma C^* \gamma^* (P^\perp - A)$ and $C' = (P^\perp - A^*)\gamma^* C \gamma (P - A^*)$, then $(C'^*, C') \in V_{(X',Y')}$, and if C^* and C are either symmetric or skew-symmetric relative to ω, then C'^* and C' are symmetric or skew-symmetric relative to $\omega_{(A^*,A)}$.

Proposition 2.5 suggests that there is considerable freedom in the choice of a pair from $V_{(X,Y)}$ to represent a given nondegenerate 2-form. In fact, Proposition 2.9 shows that certain nondegenerate 2-forms can be represented in terms of elements of $V_{(X,Y)}$ that are skew-symmetric. To do this, it is necessary to consider triples of transverse pairs of splittings of V. In these arguments the following generalization of the addition law of velocity is required. The proof is an exercise in linear algebra.

Proposition. 2.7. Given splittings (X, Y), (X', Y'), and (X'', Y'') such that $((X, Y), (X', Y'))$, $((X, Y), (X'', Y''))$, and $((X', Y'), (X'', Y''))$ are transverse pairs with graph coordinates (C^*, C), (A^*, A), and (A'^*, A'), respectively, then

$$(A-C)P' = (P^{\perp} - AC^*)A' \tag{5}$$

$$(A^* - C^*)P'^{\perp} = (P - A^*C)A'^* \tag{6}$$

where P is the projection determined by (X, Y) and P' is the projection determined by (X', Y').

The next proposition extends Proposition 2.5. The proof uses Proposition 2.5 and the fact that two elements of $D_{(X,Y)}$ are related by (3). This proposition is used to show that the construction of Proposition 2.9 gives the desired 2-form.

Proposition 2.8. Let (A^*, A), $(A'^*, A') \in V_{(X,Y)}$, and $(C_1^* C) \in V_{(X',Y')}$ be the graph coordinates of the transverse pair of splittings $((X, Y), (X', Y'))$, $((X, Y), (X'', Y''))$, and $((X', Y'), (X'', Y''))$, respectively. For any $\omega \in D_{(X,Y)}$, $\omega_{(A^*,A)} = \omega_{(A'^*,A')}$ if and only if C^* and C are symmetric relative to $\omega_{(A^*,A)}$.

Proof. If (C^*, C) is symmetric relative to $\omega_{(A^*,A)}$, then (X'', Y'') is a Lagrangian pair for $\omega_{(A^*,A)}$, and so there is $E \in M_{(X,Y)}$ such that $\omega_{(A^*,A)} = (\omega_E)_{(A'^*,A')}$. Choose $\Lambda \in D^*_{(X,Y)}$ so that $\mathscr{C}(\Lambda, \omega) = I$. Then Proposition 2.4 and this identity imply that

$$\begin{aligned}&\mathscr{C}((\Lambda_E)_{(A'^*,A')}, (\omega_E)_{(A'^*,A')})\\&\quad = (P^{\perp} + A'^*)\gamma'^* E^{t2} \gamma'^* (P - A'^*)\\&\qquad + (P + A')\gamma' E^2 \gamma' (P^{\perp} - A') = P''^{\perp} + P''\end{aligned}$$

and so $E^2 = P$. Clearly, E is a continuous function of (C^*, C), and $V_{(X',Y')}$ is connected. Therefore, the fact that for $(C^*, C) = (0, 0)$, $E = P$ and the fact that the identity is isolated in the set of idenipotent transformations imply that $E = P$.

Conversely, suppose that $\omega_{(A^*,A)} = \omega_{(A'^*,A')}$. Since (X'', Y'') is a Lagrangian pair for $(\omega_{(A^*,A)})_{(C^*,C)}$ and $\omega_{(A'^*,A')}$, there exists $E \in M_{(X,Y)}$ such that $((\omega_E)_{(A^*,A)})_{(C^*,C)} = \omega_{(A'^*,A')}$. But then (X', Y') is a Lagrangian pair for $((\omega_E)_{(A^*,A)})_{(C^*,C)}$ and so Corollary 2.1 implies that

$((\omega_E)_{(A^*,A)})_{(C^*,C)} = (\omega_E)_{(A^*,A)}$. This in turn implies that $(\omega_E)_{(A^*,A)} = \omega_{(A^*,A)}$ and so $\omega_E = \omega$. Hence, C^* and C are symmetric relative to $\omega_{(A^*,A)}$. ■

Proposition 2.9 gives the main result of this section. It shows that given $\omega \in D_{(X,Y)}$ if $(A^*, A) \in V_{(X,Y)}$ satisfies certain conditions, then $\omega_{(A^*,A)}$ can be represented by $\omega_{(A'^*,A')}$, where A'^* and A' are skew-symmetric relative to ω. As a tool to be used in the proof, introduce a complex structure $J \in \mathrm{End}(V)$ such that (1) $J: X \to Y$ and (2) for $u, v \in Y$, $g(u, v) = \omega(u, Jv)$ is an inner product on Y.

Proposition 2.9. Let $(A^*, A) \in V_{(X,Y)}$ be the graph coordinate of the transverse pair $((X, Y), (X', Y'))$ and let $\omega \in D_{(X,Y)}$. If a superscript t denotes the transpose relative to ω, and P is the projection determined (X, Y), then define $H = \gamma^t(A^t - A)\gamma$, $K = \gamma^{*t}(A^{*t} - A^*)\gamma^*$, and $L = \gamma^{*t}(P - A^{*t}A)\gamma$. If (1) $\ker(H) = X$ and $\ker(K) = Y$, (2) $\mathrm{spec}(KL^tK^{\#}L + KH) \subset \mathbb{C} - (0, \infty)$ and $\mathrm{spec}(HLH^{\#}L^t + HK) \subset \mathbb{C} - (0, \infty)$, and (3) $L - L^t$ is sufficiently small, then there is a pair $(A'^*, A') \in V_{(X,Y)}$ such that (A'^*, A') are skew-symmetric and $\omega_{(A^*,A)} = \omega_{(A'^*,A')}$.

Proof. The pair (A'^*, A') will be constructed from a transverse pair $((X, Y), (X'', Y''))$ with the property that the graph coordinate (C'^*, C') of $((X', Y'), (X'', Y''))$ is symmetric relative to $\omega_{(A^*,A)}$ and the graph coordinate (A'^*, A') is skew-symmetric. Suppose that $(C'^*, C') \in V_{(X',Y')}$ is determined from $(C^*, C) \in V_{(X,Y)}$ using Proposition 2.6. Then, using Proposition 2.7, we find that the graph coordinate (A'^*, A') of $((X', Y'), (X'', Y''))$ is given by

$$A'^* = (A^* + \gamma C^* \gamma^{**})(P^{\perp} + A\gamma C^* \gamma^{**})^{\#}$$

$$A' = (A + \gamma^* C \gamma^{\#})(P + A^* \gamma^* C \gamma^{\#})^{\#}$$

Assuming that C^* and C are symmetric, (A'^*, A') will be skew-symmetric if

$$-CKC + CL + L^tC - H = 0 \tag{7}$$

$$-C^*HC^* + LC^* + C^*L^t - K = 0 \tag{8}$$

We solve (7), as the solution to (8) is similar. Note that if a superscript ℓ denotes the transpose relative to g, then $L^t = -JL^{\ell}J$, and so multiplying (7) by J gives

$$JC(KJ)JC + JCL + L^{\ell}JC - JH = 0 \tag{7$'$}$$

Since $KJ \in \mathrm{End}(Y)$ is symmetric relative to g, there is $\sigma \in \mathrm{End}(Y)$ such that $\sigma^2 = 1$, $\sigma^{\ell} = \sigma$ and there is $k \in \mathrm{End}(Y)$ such that k is positive-definite relative to g and such that $KJ = k\sigma k$. Let $j \in \mathrm{End}(Y \otimes \mathbb{C})$ such that $j^2 = \sigma$, $\bar{j}j = 1$, and $j^{\ell} = j$. Multiplying (7′) by jk on the left and kj on the right, letting

$L' = j^{-1}k^{-1}Lkj$ and $H' = jkJHkj$, and setting C equal to $jkCkj$, we find that (7′) becomes

$$(C+L')^t(C+L') = L'^tL' + H' \qquad (7'')$$

Let $T = L'^tL' + H'$. Note that T is real relative to the conjugation defined by σ, i.e., $\sigma\bar{T}\sigma = T$. Since T is similar to $-(KL^tK^{\#}L + KH)$, (2) implies that there is a real interpolating polynomial $g(z)$ for the standard branch of $z^{1/2}$ on the spectrum of T. Let $S = g(T)$. Then $S^2 = T$, S is real relative to σ, and S is symmetric. Let $O(Y, \mathbb{C})$ be the complex orthogonal group on $Y \otimes \mathbb{C}$ defined by g and let $O_\sigma = \{o \in O(Y, \mathbb{C}) | \sigma\bar{o}\sigma = o\}$. Also let

$$G_\sigma = \{h \in \mathrm{End}(V \otimes \mathbb{C}) \,|\, \sigma\bar{h}\sigma = h \text{ and } h^t = -h\}$$

Equation (7″) will have a solution if there is $o \in O_\sigma$ such that $So - o^tS = L' - L'^t$. However, the map $\alpha \cdot O_\sigma \to G_\sigma$, defined by $\alpha(o) = So - o^tS$, is invertible at the identity. This follows from the fact that if $g(z)$ is determined by the standard branch $f(z) = z^{1/2}$, then $\mathrm{spec}(S) \cap \mathrm{spec}(-S) = \Phi$. ■

Proposition 2.9(3) states that the conditions guaranteeing the existence of a skew-symmetric pair (A'^*, A') such that $\omega_{(A^*,A)} = \omega_{(A'^*,A')}$ are open. The formulation of a sufficient condition to replace (3) is complicated. Note that since O_σ is isomorphic to $O(p, q)$ for integers p and q with $p + q = n$, and since $O(p, q)$ has compact subgroups, it is clear that for some $L'^t - L'$ sufficiently large, solutions to (7″) will not exist. Proposition 2.9(2) is too strong, for it is possible in certain circumstances to solve (7″) for real endomorphism even though T has negative eigenvalues. For instance, if n is even and all real eigenvalues of T are negative, the solutions to (7″) always exist. Also, in practice, it is necessary to solve (7) and (8) without the assumption that K and H are nonsingular on Y and X. When Proposition 2.9(1), is relaxed (7) and (8) become a coupled pair of matrix equations. The solution of these equations is not as easy as in the nonsingular case.

Proposition 2.9 is crucial to the arguments of the next section. This is because the following result holds for pairs from $V_{(X,Y)}$ that are skew-symmetric relative to some $\omega \in D_{(X,Y)}$.

Proposition 2.10. Let $((X, Y), (X', Y'))$, $((X', Y'), (X'', Y''))$, and $((X, Y), (X'', Y''))$ be transverse pairs of splittings with graph coordinates (A^*, A), (C^*, C), and (A'^*, A'). If $\omega \in D_{(X,Y)}$ and if A^*, A, A'^*, and A' are skew-symmetric relative to ω, then $(\omega_{(A^*,A)})_{(C^*,C)} = \omega_{(A'^*,A')}$.

3. IMPLICATIONS FOR DYNAMICS

The general construction mentioned in the introduction gives a second-order dynamical formalism for sections of a fibered manifold pair (M, S, π).

Here S is the base and $\pi: M \to S$ is the fibration. The idea is to extend sections of (M, S, π) to sections of $\Lambda^k(M)$ for some appropriate choice of k. On $\Lambda^k(M)$ there is a natural generalization of a Hamiltonian structure that determines when an extended section is a solution to the given dynamical conditions. The first example of this construction is Hamiltonian mechanics, where $S = \mathbb{R}$, M is arbitrary, and $k = 1$. In this instance the dynamical conditions are specified by a Hamiltonian vector field on T^*M. The following discussion shall examine the case where S is arbitrary, $M = T^*S$, and $k = 2$. It shall be argued that these dynamical structures can be associated with electrodynamics. For simplicity assume that $S = \mathbb{R}^n$ and so $M = T^*\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ where $\pi: T^*\mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the first factor.

Begin by transferring in the standard manner the linear algebraic structures developed in Section 2 to manifolds. If X is a subbundle of $TT^*\mathbb{R}^n$ and $z \in T^*\mathbb{R}^n$, denote by X_z the subspace of $TT^*\mathbb{R}^n_z$ determined by X at z. Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be the standard coordinates on $T^*\mathbb{R}^n$ and let $(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ be the standard coordinate vector fields. Relative to these coordinates the canonical symplectic and cosymplectic structures on $T^*\mathbb{R}^n$ are determined by the 2-form $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$ and the 2-vector field $\Lambda_0 = \sum_{i=1}^n P_i \wedge Q_i$. $TT^*\mathbb{R}^n$ possesses a natural splitting (X_0, Y_0) given by $X_{0z} = 0 \times \mathbb{R}^n$ and $Y_{0z} = \mathbb{R}^n \times 0$. Clearly, $\omega_0 \in D_{(X_0, Y_0)}$ and $\Lambda_0 \in D^*_{(X_0, Y_0)}$. Let P_0 be the projection determined by (X_0, Y_0). If (X, Y) is a second constant splitting of $TT^*\mathbb{R}^n$ such that $((X_0, Y_0), (X, Y))$ is a transverse pair with graph coordinate $(A^*, A) \in V_{(X_0, Y_0)}$, and if $E \in M^t_{(X_0, Y_0)}$ is also constant, then $(\omega_{0E})_{(A^*, A)}$ is closed, and a natural choice of Darboux coordinates is given by

$$(q', p') = (\gamma(Eq - A^*p), \gamma^*(p - AEq)) \tag{9}$$

Note that if $A^* = 0$, if A is skew-symmetric relative to ω_0, and if $E = P_0$, then (9) reduces to the Legendre transformation determined by the constant field $2A$.

The principal dynamical objects of this construction are submanifolds $\Gamma \subset T^*\mathbb{R}^n$ such that $\pi|_\Gamma$ is a diffeomorphism. These objects represent unparametrized vector potentials. The dynamics of Γ depends on the choice of a *parametrization* of Γ by a diffeomorphism $\sigma: \mathbb{R}^n \to \Gamma$, $\sigma = (\sigma_1, \sigma_2)$.

Definition 3.1. A *momentum* for σ is a map $\lambda_\sigma: \mathbb{R}^n \to \Lambda^2(T^*\mathbb{R}^n)|_\Gamma$ such that $\lambda_\sigma(q) \in \Lambda^2(T^*\mathbb{R}^n)_{\sigma(q)}$. Equivalently, λ_σ is a section of $\sigma^*\Lambda^2(T^*\mathbb{R}^n)$.

The first step in constructing a dynamics for Γ is to introduce an algorithm that determines a momentum for a given parametrization σ. The algorithm to be introduced is similar to the procedure given in special relativity to determine the momentum of a parametrized world line $\gamma: \mathbb{R} \to \mathbb{R}^4$. Recall that the momentum of γ at $t_0 \in \mathbb{R}$ can be determined by boosting the standard coordinates to the rest frame of γ at t_0. The time component of $\dot{\gamma}(t_0)$ determines the mass, the boost determines the velocity, and these quantities in turn determine the momentum. An analogous procedure can be implemented here. If $f:\mathbb{R}^n \to \mathbb{R}^m$, denote the Jacobian of f by Df.

Proposition 3.1. Let $\sigma:\mathbb{R}^n \to T^*\mathbb{R}^n$ be such that $\pi \circ \sigma$ is a diffeomorphism. If for $q \in \mathbb{R}^n$ and $z = \sigma(q)$, $(A^*, A) \in V_{(X_{0z}, Y_{0z})}$ and $E \in M_{(X_{0z}, Y_{0z})}$ satisfy (1) $D\sigma_2(q) - AD\sigma_1(q) = 0$ and (2) $\gamma ED\sigma_1(q) = 1$, then $D(q' \circ \sigma(q)) = 1$ and $D(p' \circ \sigma(q)) = 0$.

Proof. The proof is a computation using (9) and the chain rule. ∎

Proposition 3.1 states that if $(A^*, A) \in V_{(X_0, Y_0)}$ and $E \in M_{(X_0, Y_0)}$ are constant and satisfy hypotheses (1) and (2) at $q \in \mathbb{R}^n$, then σ agrees up to first order at q with a map σ' constant in the coordinates (9), i.e., $\sigma' = (q', c)$. In this case it can be said that σ is instantaneously at rest at q in the coordinates (9). Let the pair (A^*, A) be called the *velocity* of σ at q, let E be called the *charge* of σ at q, and let $\lambda_\sigma:\mathbb{R}^n \to \sigma^*\Lambda^2(T^*\mathbb{R}^n)$ defined by $\lambda_\sigma(q) = (\omega_{0E})_{(A^*, A)}$ be called the *relativistic momentum* of σ at q. To justify this choice of terms, note that A is the graph coordinate of $T\Gamma_z$ relative to the splitting (X_{0z}, Y_{0z}) of $TT^*\mathbb{R}^n_z$, and so gives projective first-order information about the vector potential, in the same way that the velocity gives projective information about the tangent to a parametrized world line. If $A^* = 0$ and $E = P_0$, then σ is a section of $T^*\mathbb{R}^n$ and the relativistic momentum reduces to the field strength of the vector potential σ. Also, if $\sigma_1(q) = cq$, then $\sigma_2(q) = (\pi|_\Gamma)^{-1}(cq)$, and so if $E = cP_0$, the vector potential must be scaled by c. Hence, E can be identified with a scaling of the charge of the vector potential much in the same way that a scaling of the time parameter can be identified with the scaling of the mass associated with a parametrized world line.

The parametrization σ does not uniquely determine the velocity. Since $D\sigma_1$ is invertible, A is determined by (1), but A^* must be determined in terms of A by a constitutive relation. The standard constitutive relation of free space can be implemented by introducing an almost complex structure $J \in \mathrm{End}(T^*\mathbb{R}^n)$ such that (1) J is symmetric relative to ω_0, (2) $JX_0 = Y_0$, and (3) for $u, v \in TT^*\mathbb{R}^n_z$, $g(u, v) = \omega_0(u, Jv)$ is a Minkowski inner product. To determine A^* from A, set $A^* = JA^tJ$, where the transpose is determined by ω_0. Let $V^c_{(X_0, Y_0)} \subset V_{(X_0, Y_0)}$ be the elements of $V_{(X_0, Y_0)}$ that satisfy this equality.

Now we describe the construction that produces dynamical equations for parametrizations of Γ. To apply this procedure, it is necessary to extend the momentum $\lambda_\sigma \circ \pi|_\Gamma:\Gamma \to \Lambda^2(T^*\mathbb{R}^n)|_\Gamma$ to a section of $\Lambda^2(T^*\mathbb{R}^n)$ defined in a neighborhood of Γ. To accomplish this, let Z be a subbundle of $TT^*\mathbb{R}^n$ of dimension n transverse to Γ; i.e., for all $z \in \Gamma$, $TT^*\mathbb{R}^n_z = T\Gamma_z \oplus Z_z$. For each $z \in \Gamma$, if Z_z is viewed as a linear submanifold $T^*\mathbb{R}^n$, then extend $\lambda_\sigma \circ \pi|_\Gamma$ along Z by affine translation. This procedure constructs a section $\omega_\sigma: U \to \Lambda^2(T^*\mathbb{R}^n)$, for some neighborhood U of Γ, such that $\omega_\sigma|_\Gamma = \lambda_\sigma \circ \pi|_\Gamma$ and if X is an affine vector field on U along Z, then $L_X\omega_\sigma = 0$. In the standard model of electrodynamics Z is taken to be the vertical distribution; see Souriau (1970).

The dynamics of σ is determined by ω_σ and a dynamical structure on $\Lambda^2(T^*\mathbb{R}^n)$ that is analogous to the Minkowski Hamiltonian structure on $T^*\mathbb{R}^n$. Adopt the convention that underscored indices are associated with p coordinates and indices that are not underscored are associated with q coordinates. Using this notation, let $(q_i, p_i, p_{ij}, p_{\underline{i}\underline{j}}, p_{\underline{i}j})$ be the natural coordinates on $\Lambda^2(T^*\mathbb{R}^n)$; i.e., if $\omega \in \Lambda^2(T^*\mathbb{R}^n)_{(q,p)}$, then

$$\omega = \sum p_{ij}\, dq_i \wedge dq_j + \sum p_{\underline{i}\underline{j}}\, dp_i \wedge dp_j + \sum p_{\underline{i}j}\, dp_i \wedge dq_j$$

Define the function H on $\Lambda^2(T^*\mathbb{R}^n)$ by

$$H = \tfrac{1}{2}\left(\sum p_{ij} J_{\underline{i}l}\, p_{lm} J_{m\underline{j}} + \sum p_{\underline{i}\underline{j}}\, J_{i\underline{l}}\, p_{\underline{l}\underline{m}}\, J_{\underline{m}j} + \sum p_{ij}\, J_{i\underline{m}}\, p_{\underline{m}l}\, J_{l\underline{j}}\right).$$

The canonical 3-form Ω on $\Lambda^2(T^*\mathbb{R}^n)$ is given by

$$\Omega = \sum dp_{ij} \wedge dq_i \wedge dq_j + \sum dp_{\underline{i}\underline{j}} \wedge dp_i \wedge dp_j + \sum dp_{\underline{i}j} \wedge dp_i \wedge dq_j$$

Note that the 2-vector field

$$\Pi = \sum J_{\underline{i}l}\, p_{lm} J_{m\underline{j}}\, Q_i \wedge Q_j + \sum J_{i\underline{l}}\, p_{\underline{l}\underline{m}}\, J_{\underline{l}j}\, P_i \wedge P_j + \sum J_{jm}\, p_{\underline{m}l}\, J_{l\underline{i}}\, P_i \wedge Q_i$$

has the property $i(\Pi)\Omega = dH$. Therefore, Π plays the role of a Hamiltonian 2-vector field for the function H. Although, unlike Hamiltonian 1-vector fields, Π is not uniquely determined by H and Ω, but also depends on a choice of horizontal. As in Hamiltonian mechanics, Π induces a Legendre transform $\mathscr{L}$, which in this case maps 2-forms on $T^*\mathbb{R}^n$ into 2-vectors on $T^*\mathbb{R}^n$. If ω is a 2-form and $z \in T^*\mathbb{R}^n$, define $\mathscr{L}(\omega)(z) = \pi_*\Pi_{\omega(z)}$, where π is the projection of $\Lambda^2(T^*\mathbb{R}^n)$ onto $T^*\mathbb{R}^n$. The following proposition is a direct consequence of Definition 2.1.

Proposition 3.2. Let ω be a 2-form on $T^*\mathbb{R}^n$. If, for $z \in T^*\mathbb{R}^n$, there is $(A^*, A) \in V^c_{(X_{0z}, Y_{0z})}$ and $E \in M_{(X_{0z}, Y_{0z})}$ such that $\omega(z) = (\omega_{0E})_{(A^*,A)}$, then $\mathscr{L}(\omega)(z) = (\Lambda_{0JEtJ})_{(A^*,A)}$.

By a generalization of Hamilton–Jacobi theory, the 2-vector field Π induces a set of first-order partial differential equations for sections of $\Lambda^2(T^*\mathbb{R}^n)$. It is found that the pair $(\mathscr{L}(\omega), \omega)$, determined by a 2-form ω on $T^*\mathbb{R}^n$, is a solution to the dynamical problem determined by Π if the 1-form $f_\omega = i(\mathscr{L}(\omega))\, d\omega - \tfrac{1}{2}d(i(\mathscr{L}(\omega)\omega)$ vanishes. We call f_ω the *force* on ω. Note the similarity between this expression and the intrinsic expression of the geodesics equation in metric geometry.

Now suppose that σ is chosen so that the charge of σ is the identity; i.e., $E = P_0$. In this case Proposition 2.4 implies that $\mathscr{C}(\mathscr{L}(\omega_\sigma), \omega_\sigma) = \mathscr{C}(\Lambda_0, \omega_0) = I$, and so the expression for the force reduces to

$$f_{\omega_\sigma} = i(\mathscr{L}(\omega_\sigma))\, d\omega_\sigma \tag{10}$$

One difference between this equation and the mechanical analog is that in this case the expression for the force depends upon how the momentum λ_σ is extended to a neighborhood of Γ, while in the mechanical case the force depends only on the tangent vector to the curve in question. The next proposition shows how the choice of the subbundle Z affects quantities used in the calculation of (10). First note that the coordinate vector fields (Q_i', P_i') determined by (9) with $E = P_0$ are given by

$$(Q_i', P_i') = ((P_0 + A)\gamma Q_i, (P_0^\perp + A^*)\gamma^* P_i) \tag{11}$$

The following arguments require that a graph coordinate (C^*, C) relative to a splitting (X, Y) be expressed in terms of their components relative to the frame field given by (11). Adopt the convention that the indices of the matrix representation $(\{C_{ij}^*\}, \{C_{ij}\})_{1\le i,j\le n}$ for (C^*, C), refer to the frame field specified by choosing the frame at $z \in T^*\mathbb{R}^n$ to be given by (11). Here (A^*, A) is the graph coordinate of the transverse pair $((X_{0z}, Y_{0z}), (X_z, Y_z))$. Let $B^* \in \mathrm{End}(T^*\mathbb{R}^n)$, $B^*: X_0 \to Y_0$, be the graph coordinate of Z relative to the splitting (X_0, Y_0).

Proposition 3.3. Let $((X_0, Y_0), (X, Y))$ be a transverse pair of splittings with graph coordinate $(A^*, A) \in V_{(X_0, Y_0)}$. At $z \in T^*\mathbb{R}^n$, let $B^* \in \mathrm{End}(T^*\mathbb{R}^n)_z$ such that $\mathrm{ran}(B^*) \subset Y_{0z} \subset \ker(B^*)$ and for all $i, j, k \in (1, \ldots, n)$, $(P_k + B^* P_k) A_{ij}(z) = 0$ and $(P_k + B^* P_k) A_{ij}^*(z) = 0$. Let (C^*, C) be the graph coordinate of the constant splitting $((X_0, Y_0), (X', Y'))$ such that $X_z' = X_z$ and $Y_z' = Y_z$. If (A'^*, A') is the graph coordinate of $((X', Y'), (X, Y))$, then at z

$$\begin{aligned} A'^*_{\underline{i}j,k} &= (P_0 - C^*C)^{\#1/2}_{ih} A^*_{h\underline{l},m} (P_0^\perp - CC^*)^{\#1/2}_{\underline{l}j} \\ &\quad \times ((P_0 - B^*C)(P_0 - C^*C)^{\#1/2})_{mk} \end{aligned} \tag{12}$$

$$\begin{aligned} A'_{\underline{i}j,k} &= (P_0^\perp - CC^*)^{\#1/2}_{\underline{i}h} A_{\underline{h}l,m} (P_0 - C^*C)^{\#1/2}_{\underline{l}j} \\ &\quad \times ((P_0 - B^*C)(P_0 - C^*C)^{\#1/2})_{mk} \end{aligned} \tag{13}$$

$$\begin{aligned} A'^*_{\underline{i}j,\underline{k}} &= (P_0 - C^*C)^{\#1/2}_{ih} A^*_{h\underline{l},m} (P_0^\perp - CC^*)^{\#1/2}_{\underline{l}j} \\ &\quad \times ((B^* - C^*)(P_0^\perp - CC^*)^{\#1/2})_{m\underline{k}} \end{aligned} \tag{14}$$

$$\begin{aligned} A'_{\underline{i}j,\underline{k}} &= (P_0^\perp - CC^*)^{\#1/2}_{\underline{i}h} A_{\underline{h}l,m} (P_0 - C^*C)^{\#1/2} \\ &\quad \times ((B^* - C^*)(P_0^\perp - CC^*)^{\#1/2})_{m\underline{k}} \end{aligned} \tag{15}$$

Proof. First note that if X is a vector field such that $L_X P_0 = 0$, then since $A(z) = C(z)$, Proposition 2.7 implies that

$$L_X P_0^{\perp} A'(z) = (P_0^{\perp} - CC^*)^{\#} L_X A(z)$$
$$L_X P_0 A'(z) = C^*(P_0^{\perp} - CC^*)^{\#} L_X A(z)$$

So $L_X A(z) = 0$ implies $L_X A'(z) = 0$. Therefore, for $i, j, k \in (1, \ldots, n)$,

$$(P_k - B^* P_k) A'_{ij}(z) = 0$$

Now

$$A'_{ij} = -\omega_{0(A^*, A)}(Q'_i, A'Q'_j)$$

But

$$A'Q'_i = (P_0^{\perp} + C^*)(P_0^{\perp} - AC^*)^{\#}(A - C)(P_0 - C^*C)^{\#1/2} Q_i$$

and so

$$A'_{ij} = \omega_0((P_0^{\perp} - CC^*)^{1/2}(P_0^{\perp} - AC^*)^{\#}(A - C)(P_0 - C^*C)^{\#1/2} Q_j, Q_i)$$

Differentiating with respect Q_k and evaluating at z gives

$$Q_k A'_{ij}(z) = (P_0^{\perp} - CC^*)^{\#1/2}_{\underline{i}\underline{h}} Q_k A_{\underline{h}l}(z)(P_0 - C^*C)^{\#1/2}_{\underline{l}j}$$

But (11) implies that

$$Q_k = (Q'_l - P'_i C_{\underline{i}l})(P_0 - CC^*)^{\#1/2}_{lk}$$

and so

$$Q_k A'_{ij}(z) = Q'_l A'_{ij}(z)((P_0 - C^*C)^{1/2}(P_0 - B^*C)^{\#})_{lk}$$

Substituting this expression gives (12). The rest follow by similar arguments. ■

The central result of this construction is that it is possible to choose the subbundle Z such that if λ_σ is extended affinely along Z, then the force on ω_σ in standard coordinates has the following properties. (1) if at $z \in \Gamma$, $A^*(z) = 0$, then

$$f_{\omega_\alpha} = \left(0, \sum_{i,j} (A^* + A^{*\prime})(z)_{\underline{ij}, i}\, dp_j\right)$$

and (2) if $f_{\omega_\sigma} = (f_1, f_2)$, then there is $C \in \mathrm{End}(T^*\mathbb{R}^n)$ such that C is skew-symmetric and $f_2 C = f_1$. Property (1) is the statement that for small A^*, the force determined by (10) should reduce to the standard force law of electromagnetism. Actually, the reduction to the standard model does not depend on the choice of Z, but is a consequence of Definition 2.1. Property

(2) is the stronger requirement. As suggested by property (1), the p component of the force is related to the field equations. Property (2) requires that the q component of the force be linearly related to the p component in a manner that is analogous to the relation between force and power in special relativity. Property (2) will imply that in the limit of small A^* the q component of the force reduces to the divergence of the trace-free electromagnetic energy-momentum tensor.

To choose the subbundle Z such that ω_σ satisfies (1) and (2), it is necessary to assume that if $\lambda_\sigma = \omega_{0(A^*, A)}$, where A is the graph coordinate of $T\Gamma$ relative to (X_0, Y_0), then there is a skew-symmetric pair (A'^*, A') so that $\lambda_\sigma = \omega_{0(A'^*, A')}$. Thus, it shall be assumed that the conclusions of Proposition 2.9 apply to the pair (A^*, A). If, along Γ, λ_σ is extended in the direction determined by A'^*, then the force will satisfy (1) and (2) with $C = A'$.

Proposition 3.4. Let $(A^*, A) \in V^c_{(X_0, Y_0)}|_\Gamma$ such that there exists $(A'^*, A') \in V_{(X_0, Y_0)}|_\Gamma$ with A'^* and A' skew-symmetric relative to ω_0 and $\omega_{0(A^*, A)} = \omega_{0(A'^*, A')}$. If at $z \in \Gamma$, $B^*(z) = A'^*(z)$, then the force at z is given by

$$f_{\omega_{0(A^*, A)}} = 2 \sum_k \sum_{i,l} A'^*_{\underline{il}, i}(P_0^\perp - A'^*A')^{\#}_{\underline{l}k}\, dp_k - 2 \sum_k \sum_{i,l} A'^*_{\underline{il}, i} A'_{\underline{l}j}(P_0 - A'^*A')^{\#}_{jk}\, dq_k \tag{16}$$

Proof. Let (X', Y') be the splitting determined by (A'^*, A') and let (X'', Y'') be the constant splitting such that $X''(z) = X'(z)$ and $Y''(z) = Y'(z)$. Let (C^*, C) be the graph coordinate of $((X_0, Y_0), (X'', Y''))$ and let (A''^*, A'') be the graph coordinate of $((X'', Y''), (X', Y'))$. If $\mu = \omega_{0(C^*, C)}$, then relative to the coordinates (9) determined by $((X_0, Y_0), (X'', Y''))$ with $E = P_0$ it is easy to see that

$$f_{\mu_{(A''^*, A'')}}(z) = 2 \sum_{k=1}^{n} \left(\sum_{i=1}^{n} A''^*_{\underline{ik}, i}(z) \right) dp'_k + 2 \sum_{k=1}^{n} \left(\sum_{i=1}^{n} A''_{\underline{ik}, \underline{i}}(z) \right) dq'_k$$

If $B^*(z) = A''^*(z)$, then (15) implies that the second term in the sum vanishes at z, and (12) implies that the first term reduces to

$$f_{\mu_{(A''^*, A'')}}(z) = 2 \sum_{k=1}^{n} \left(\sum_{i,l=1}^{n} A'^*_{\underline{il}, i}(z)(P_0^\perp - A'A'^*)^{\# 1/2}_{\underline{l}k}(p) \right) dp'_k$$

Now substitute

$$dp'_k = (P_0^\perp - A'A'^*)^{\# 1/2}_{\underline{k}l}\, dp_l - (P_0^\perp - A'A'^*)^{\# 1/2}_{\underline{k}j} A'_{\underline{j}l}\, dq_l$$

for dp'_k, to get

$$f_{\mu_{(A''^*, A'')}} = 2 \sum_k \sum_{i,l} A'^*_{\underline{il}, i}(P_0^\perp - A'^*A')^{\#}_{\underline{l}k}\, dp_k - 2 \sum_k \sum_{i,l} A'^*_{\underline{il}, i} A'_{lj}(P_0 - A'^*A')^{\#}_{jk}\, dq_k$$

But Proposition 2.10 implies that $\mu_{(A''^*, A'')} = \omega_{0(A'^*, A')} = \omega_{0(A^*, A)}$ and similarly if $M = \Lambda_{0(C^*, C)}$, then $M_{(A''^*, A'')} = \Lambda_{0(A'^*, A')} = \Lambda_{0(A^*, A)}$. So, by Proposition 3.2, $f_{\mu(A''^*, A'')} = f_{\omega_0(A^*, A)}$. ■

Proposition 3.4 shows that Proposition 2.9 extends the standard relation between the derivative of the vector potential and the field. In fact, if $A^* = 0$, it is easy to see that the solution to (7) is the skew-symmetric part of A. However, for $A^* \neq 0$, Proposition 2.9 says that in order to guarantee a solution to (7) and (8), A must be bounded. This has the interesting consequence that in this model of electrodynamics the conservation laws may break down at sufficiently high field strengths.

4. CONCLUDING REMARKS

The first remark is that Proposition 3.4 is the result advertised in the introduction. The crucial assumption in the derivation of (16) is that the relativistic momentum λ_σ of the parametrization σ of Γ must satisfy $\mathscr{C}(\mathscr{L}(\lambda_\sigma), \lambda_\sigma) = cI$. Proposition 3.4 shows that by restricting the value of the charge, one obtains a relation between the p and q components of the dynamical 1-form f_{ω_σ} that is analogous to the relation between power and force that follows from the conservation of mass in relativistic dynamics. One interesting and perhaps important characteristic of this construction is that the quantity that appears to represent charge is not a scalar quantity, but rather is endomorphism-valued. Also note that when the charge is restricted to scalar values, one obtains a "relativistic" analog of classical electrodynamics.

Returning to the proof of Proposition 3.4, if

$$f_{\mu(A''^*, A'')} = 2 \sum_{i=1}^{n} J_i \, dp_i'$$

then after transforming to standard coordinates, one can write (16) equivalently as

$$\sum_i A'^*_{ik,i} = \sum_l J_l (P_0^\perp - A'^* A')^{1/2}_{lk} \tag{17}$$

$$\sum_{i,l} A'^*_{il,i} A'_{lk} = \sum_{l,j} J_l (P_0^\perp - A'^* A')^{1/2}_{lj} A'_{jk} \tag{18}$$

Note that if one neglects factors of γ, then (17) reduces to the dynamical Maxwell equations, and (18) along with the condition $A'_{(ij,k)} = 0$ is the divergence law for the energy-momentum tensor.

Solutions to (17) give those vector potentials Γ that are generated by the current J_i. However, the solution of (16) is complicated by the manner in which (A'^*, A') depend on the parametrization of Γ. A simpler but somewhat nonphysical model can be obtained by choosing the distribution Z to be the vertical distribution. In this model a system of equations similar to (17) can be shown to have bounded, radially symmetric electrostatic solutions. The scale of these solutions is determined by the boundary condition at infinity and a constant g that equates the physical dimensions of A^* and A; i.e., $A^* = (1/g^2)JA^tJ$. In this discussion g has been set to 1. The constant g is the analog of the speed of light. It is not hard to see that g has the dimensions of force. If g is assumed to be a product of the fundamental constants G, c, and h, then $g = c^4/G$. As a result, the radius of the electrostatic solutions for this value of g is much smaller than the classical electron radius; in fact, it is approximately equal to the Planck length.

REFERENCES

Born, M., and Infeld, L. (1934). Foundations of a new field theory, *Proceedings of the Royal Society Series A*, **144**, 425-451.

Souriau, J. M. (1970). *Structures des Systèmes Dynamiques*, Dunod, Paris.

Sternberg, S., and Guillemin, V. (1984). *Symplectic Techniques in Physics*, Cambridge University Press, New York.

Weinstein, A. (1977). *Lectures on Symplectic Manifolds* (Conference Board of Mathematical Sciences, Vol. 29), AMS, Providence, Rhode Island.